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Identification of Mode-shapes and Eigen-frequencies of Bi-hinge Beam with Distributed Mass and Stiffness

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Abstract: In the present paper, an equivalent Three Degree of Freedom (DoF) system of a bi-hinge beam, which has infinity number of degree of freedoms because possesses distributed mass and stiffness along its length, is presented. Based on the vibration partial differential equation of the abovementioned bi-hinge beam, an equivalent, mathematically, three-degree of freedom system, where the equivalent mass matrix is analytically formulated with reference on specific mass locations. Using the Three DoF model, the first three fundamental mode-shapes of the real beam can be identified. Furthermore, taking account the 3x3 mass matrix, it is possible to estimate the possible beam damages using a known technique of identification mode-shapes via records of response accelerations. Moreover, the way of instrumentation with a local network by three accelerometers is shown. It is worth noting this technique can be applied on bridges consist of bays with two hinges at its end sections, supported on elastometallic bearings, where the sense of concentrated mass is fully absent from the beam.
Keywords: Identification of mode-shapes; Distributed mass and stiffness; Continuous system.

1. Introduction

An ideal three Degree of Freedom system that is equivalent with the modal behavior of an infinity number of degree of freedom of a bi-hinges beam is presented by the present article. This equivalent three DoF system can be used in instrumentation of such beams, where the concept of the concentrated masses is not existing, with a local network of three accelerometers. This issue is a main problem that is appear very common during the instrumentation of bi-hinges bridge beams or steel stairs [1-3] or wind energy power [4-5] in order to identify the real vibration mode shapes of the structure via records of response accelerograms at specific positions due to ambient excitation [6-7].

2. Modal analysis of undamped bi-hinge beam with distributed mass and stiffness

According to the Theory of Continuous Systems [8-9], consider a straight beam that is loaded by an external continuous dynamic loading $p_z(x,t)$, with reference to a Cartesian three-dimensional reference system *oxyz*, (Figure 1). This beam possesses a distributed mass per unit length $m(x)$, which in the special case of uniform distribution is given as $m(x)=\overline{m}$ in tons per meter (tn/m). Moreover, this beam has section flexural stiffness $EI_y(x)$, where in the special case of an uniform distribution of the stiffness it is given as $EI_y(x)=EI_y$, where E is the material modulus of elasticity and I_y is the section moment of inertia about y-axis. Next. We are examining a such bi-hinges beam that possesses constant value of distributed mass along its length, as well as constant value of distributed section flexural stiffness EI_y. Due to fact that the beam mass is continuously distributed, this beam has infinity number of degrees of freedom for vibration along the vertical *oz*-axis. For the formulate of the motion equation of this beam, we consider an infinitesimal part of the beam, at location *x* from the origin *o*, that has isolated by two very nearest parallel sections. The infinitesimal length of this part is the ldx. On this infinitesimal length, we notice the flexural moment $M(x,t)$, the shear force $Q(x,t)$ with their differential increments, while the axial force $N(x,t)$ is ignored, because it doesn't affect the vertical beam vibration along z-axis. Moreover, noted the resulting force $P_z(x,t)$ of the external dynamic loading. Therefore, we can write:

$$P_z(x,t)=p_z(x,t)\cdot dx \tag{1}$$

where the resulting force $P_z(x,t)$ acts at the total beam infinitesimal part.

Furthermore, according to D'Alembert Principle, the resulting inertia force $F_a(x,t)$ is noted, where:

$$F_a(x,t) = (-\bar{m} \cdot dx) \cdot \partial^2 u_z(x,t)/\partial t^2 \quad \Rightarrow \quad F_a(x,t) = (-\bar{m} \cdot dx) \cdot \ddot{u}_z(x,t) \tag{2}$$

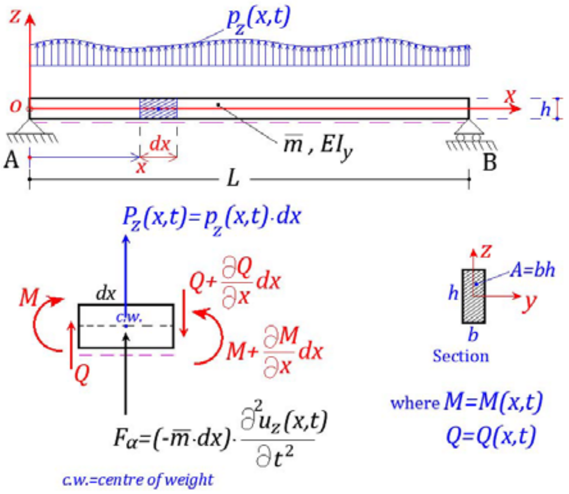

Figure 1. Bi-hinges beam with distributed mass and section flexural stiffness

Here we agree that the time derivatives of the displacements are going to symbolize with full stops, while the spatial derivatives of the displacements are going to symbolize with accent. Next, the damping and the second order differential are ignored, so the force equilibrium on the infinitesimal part of the beam along z-axis gives:

$$\sum F_z = 0 \quad \Rightarrow \quad Q + P_z(x,t) - \left(Q + \frac{\partial Q}{\partial x} dx\right) + F_a(x,t) = 0 \quad \Rightarrow$$

$$\frac{\partial Q}{\partial x} = p_z(x,t) - \bar{m} \cdot \ddot{u}_z(x,t) \tag{3}$$

Moreover, the moment equilibrium with reference to centre of weight of the infinitesimal part of the beam (see Figure 1) gives:

$$\sum M_y = 0 \quad \Rightarrow \quad M + Q \cdot \frac{dx}{2} + \left(Q + \frac{\partial Q}{\partial x} dx\right) \cdot \frac{dx}{2} - \left(M + \frac{\partial M}{\partial x} dx\right) = 0 \quad \Rightarrow \quad Q = \frac{\partial M}{\partial x} \tag{4}$$

According to Euler-Bernoulli Bending Theory (where the shear deformations are ignored) it is well-known that the following basic equation is true:

$$M(x,t) = EI_y \cdot \frac{\partial^2 u_z(x,t)}{\partial x^2} \tag{5}$$

Equation (4) and (5) are inserting into equation (3), so the motion equation without damping for the examined beam is given:

$$\frac{\partial^2 M}{\partial x^2} = p_z(x,t) - \bar{m} \cdot \ddot{u}_z(x,t) \quad \Rightarrow \quad \frac{\partial^2}{\partial x^2}\left(EI_y \cdot \frac{\partial^2 u_z(x,t)}{\partial x^2}\right) = p_z(x,t) - \bar{m} \cdot \ddot{u}_z(x,t) \quad \Rightarrow$$

$$\bar{m}\frac{\partial^2 u_z(x,t)}{\partial t^2} + EI_y \frac{\partial^4 u_z(x,t)}{\partial x^4} = p_z(x,t) \quad \Rightarrow \quad \bar{m} \cdot \ddot{u}_z(x,t) + EI_y \cdot u_z''''(x,t) = p_z(x,t) \tag{6}$$

Equation (6) is a partial differential equation that describes the motion $u_z(x,t)$ of the beam that is loaded with the external dynamic loading $p_z(x,t)$. In order to arise a unique solution from Eq.(6), the support conditions must be used at the two beam ends. It is worthy to note that the classical case of a beam with distributed mass and section flexural stiffness, under external vertical excitation (Figure 2) on the two supports is mathematically equivalent with the vibration that is described by equation (6).

Indeed, in the case of the Figure (2), the total displacement $u_z^{tot}(x,t)$ of the beam at *x*-location is given:

$$u_z^{tot}(x,t) = u_g(t) + u_z(x,t) \tag{7}$$

where $u_g(t)$ is the displacement at the base, same for the two supports.

But, it is known that the inertia forces of the beam are depended by the total displacement $u_z^{tot}(x,t)$, while the distributed dynamic loading is null, $p_z(x,t) = 0$. Thus, the equation (3) is transformed into:

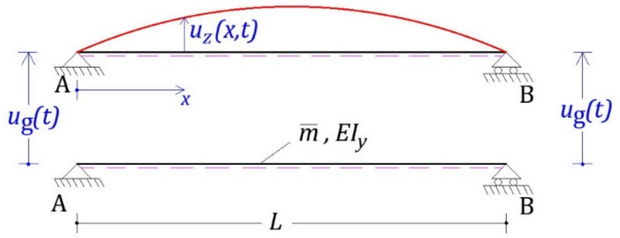

Figure 2. Beam subjected at the same vertical ground motion $u_g(t)$ on the supports

$$\frac{\partial Q}{\partial x} = p_z(x,t) - \overline{m} \cdot \frac{\partial^2 u_z^{tot}(x,t)}{\partial t^2} = 0 - \overline{m} \cdot \frac{\partial^2 u_g(t)}{\partial t^2} - \overline{m} \cdot \frac{\partial^2 u_z(x,t)}{\partial t^2} \tag{8}$$

Following, equations (4-5) are inserting into equation (8), thus we are taken:

$$\frac{\partial^2 M}{\partial x^2} = -\overline{m} \cdot \left(\frac{\partial^2 u_g(t)}{\partial t^2} + \frac{\partial^2 u_z(x,t)}{\partial t^2}\right) \qquad \Rightarrow \qquad \overline{m}\frac{\partial^2 u_z(x,t)}{\partial t^2} + EI_y \frac{\partial^4 u_z(x,t)}{\partial x^4} = -\overline{m} \cdot \frac{\partial^2 u_g(t)}{\partial t^2} \tag{9}$$

By the comparison of the two equations (6) and (9), we notice that the undamped beam vibration due to vertical motion of the two supports is mathematically equivalent with the undamped vibration of the same beam where the two supports are fixed and the beam is loaded with the equivalent distributed dynamic loading $p_{eq}(x,t)$:

$$p_{eq}(x,t) = -\overline{m} \cdot \frac{\partial^2 u_g(t)}{\partial t^2} \tag{10}$$

In the case of the beam free vibration without damping, we consider the first part of equation (9) that must be null:

$$\overline{m}\frac{\partial^2 u_z(x,t)}{\partial t^2} + EI_y \frac{\partial^4 u_z(x,t)}{\partial x^4} = 0 \tag{11}$$

Furthermore, we ask the unknown spatial-time function $u_z(x,t)$, which is the solution of equation (11), must has the form of separated variants:

$$u_z(x,t) = \varphi(x) \cdot q(t) \tag{12}$$

where $\varphi(x)$ is an unknown spatial function and the $q(t)$ is an unknown time-function.

Equation (12) has been derived two times with reference to time-dimension t and more two times with reference to spatial-dimension x, so:

$$\frac{\partial^2 u_z(x,t)}{\partial t^2} = \varphi(x) \cdot \ddot{q}(t) \quad , \quad \frac{\partial^2 u_z(x,t)}{\partial x^2} = \varphi''(x) \cdot q(t) \tag{13}$$

Equations (13) are inserting into equation (11), giving:

$$\overline{m} \cdot \varphi(x) \cdot \ddot{q}(t) + EI_y \cdot \varphi''''(x) \cdot q(t) = 0$$

and, next, divided with the numder $\overline{m} \cdot \varphi(x) \cdot q(t)$, thus we are getting:

$$\frac{-\ddot{q}(t)}{q(t)} = \frac{EI_y \cdot \varphi''''(x)}{\overline{m} \cdot \varphi(x)} \tag{14}$$

The left part of equation (14) is a time-function, but the right part is a spatial-function. In order to true equation (14) for all time values as well as for all spatial positions, the two parts of equation (14) must be equal with a constant λ. Thus, equation (14) is separated at two following differential equations:

$$\frac{-\ddot{q}(t)}{q(t)} = \lambda \quad \Rightarrow \quad \ddot{q}(t) + \lambda \cdot q(t) = 0 \tag{15}$$

$$\frac{EI_y \cdot \varphi''''(x)}{\bar{m} \cdot \varphi(x)} = \lambda \quad \Rightarrow \quad EI_y \cdot \varphi''''^{(x)} - \lambda \cdot \bar{m} \cdot \varphi(x) = 0 \tag{16}$$

However, the time equation (15) indicates a free vibration of an ideal single degree of freedom system that has eigen-frequency $\omega = \sqrt{\lambda}$. Inserting the eigen-frequensy ω into equation (16) arise:

$$EI_y \cdot \varphi''''\,(x) - \omega^2 \cdot \bar{m} \cdot \varphi(x) = 0 \quad \Rightarrow \quad \varphi''''(x) - \frac{\omega^2 \cdot \bar{m}}{EI_y} \cdot \varphi(x) = 0 \tag{17}$$

Next, we set the positive parameter β such as to be equal:

$$\beta^4 = \frac{\omega^2 \cdot \bar{m}}{EI_y}$$

because the parameters $\omega^2, \bar{m}\,, EI_y$ are always positive. By the mathematic theory it is known that the general solution of equation (17) has the following form:

$$\varphi(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \tag{18}$$

where the four unknown parameters C_1, C_2, C_3, C_4 must be calculated. In order to achieve this, four support conditions of the beam have to used. Indeed, for $x = 0\ and\ x = L$ the displacement $u_z(0,t)$ of the bi-hinge beam as well as the flexural moment $M(0,t)$, both are equal zero. The spatial function $\varphi(x)$, which is the solution of equation (18) gives the modal elastic line of the beam. Having as known data that the following equation is true:

$$\sinh \beta x = \frac{e^{\beta x} - e^{-\beta x}}{2}\,, \quad \cosh \beta x = \frac{e^{\beta x} + e^{-\beta x}}{2}$$

The spatial function of the modal elastic line for $x = 0$ is:

$$\varphi(0) = C_1 \sin 0 + C_2 \cos 0 + C_3 \sinh 0 + C_4 \cosh 0 = 0 \quad \Rightarrow \quad C_2 + C_4 = 0 \tag{19}$$

and also for $x = 0$, the function of the flexural moment due to examined modal elastic line of the beam is given by equation (5):

$$M(0,t) = EI_y \cdot \frac{\partial^2 \varphi(0)}{\partial x^2} = 0 \quad \Rightarrow \quad EI_y \cdot \varphi''(0) = 0 \tag{20}$$

Equation (18) has been derived two times with reference to spatial-dimension *x*, thus arise:

$$\varphi'\,(x) = C_1 \cdot \beta \cdot \cos \beta x + C_2 \cdot (-\beta) \cdot \sin \beta x + C_3 \cdot \beta \cdot \cosh \beta x + C_4 \cdot \beta \cdot \sinh \beta x \tag{21}$$

And

$$\varphi''(x) = C_1(-\beta^2) \cdot \sin \beta x + C_2(-\beta^2) \cdot \cos \beta x + C_3\beta^2 \cdot \sinh \beta x + C_4\beta^2 \cdot \cosh \beta x \tag{22}$$

Therefore, equation (20) is transformed:

$$EI_y \cdot \beta^2 \cdot (C_4 - C_2) = 0 \tag{23}$$

By equations (19) and (23) directly arise $C_2 = 0$ and $C_4 = 0$, thus the general solution of equation (18) is the following:

$$\varphi(x) = C_1 \sin \beta x + C_3 \sinh \beta x \tag{24}$$

In addition, the parameters C_1, C_3 are calculated by the support conditions of the second support of the beam. Therefore, for $x = L$ the vertical displacement $u(L,t) = 0$ be true. Thus, from equation (12) arise that $\varphi(L) = 0$ and equation (24) gives:

$$\varphi(L) = C_1 \sin\beta L + C_3 \sinh\beta L = 0 \tag{25}$$

In continuous, equation (20) gives:

$$M(L,t) = EI_y \cdot \frac{\partial^2\varphi(L)}{\partial x^2} = 0 \quad \Rightarrow \quad EI_y \cdot \varphi''(L) = 0 \tag{26}$$

where $\varphi''(L)$ is directly getting from equation (22) that is equivalent with zero:

$$\varphi''(L) = C_1(-\beta^2) \cdot \sin\beta L + C_3\beta^2 \cdot \sinh\beta L = 0 \tag{27}$$

However, re-written again equations (25) and (27), we are getting:

$$C_1 \cdot \sin\beta L + C_3 \cdot \sinh\beta L = 0$$
$$-C_1 \cdot \sin\beta L + C_3 \cdot \sinh\beta L = 0$$

And added part to part these two above-mentioned equations arise:

$$2 \cdot C_3 \cdot \sinh\beta L = 0 \tag{28}$$

But, the term $sinh\,\beta L$ is not equal with zero, because then vibration is not existing. Therefore, C_3 has to equal with zero, so equation (25) is formed:

$$\varphi(L) = C_1 \cdot \sin\beta L = 0 \tag{29}$$

Moreover, by equation (29) arise that either $C_1 = 0$ that is impossibility because $\varphi(x) \neq 0$ by equation (24), either $sin\,\beta L = 0$ that means the following equation must be true:

$$\beta L = n \cdot \pi \qquad n = 1,2,3,\dots \tag{30}$$

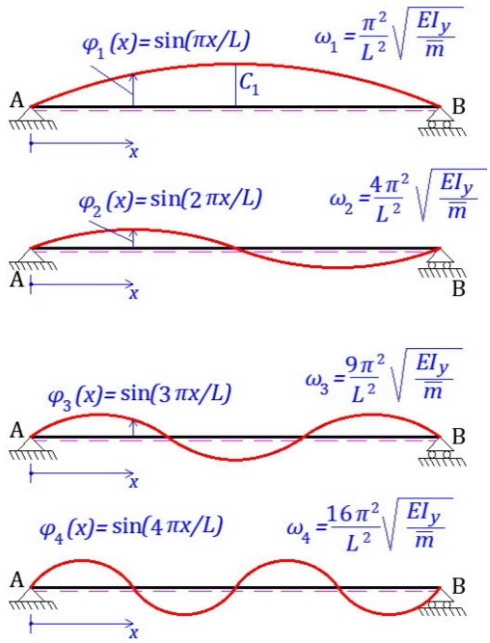

Figure 3. The four first mode-shapes of the beam with distributed mass and section flexural stiffness

However, equation (30) is transformed to equation (31):

$$\beta L = n \cdot \pi \quad \Rightarrow \quad \beta^2 L^2 = n^2 \cdot \pi^2 \quad \Rightarrow \quad \beta^2 = \frac{n^2 \cdot \pi^2}{L^2} \tag{31}$$

By the definition of parameter β , we can calculate the eigen-frequency ω:

$$\beta^4 = \frac{\omega^2 \cdot \bar{m}}{EI_y} \quad \Rightarrow \quad \omega^2 = \frac{EI_y \cdot \beta^4}{\bar{m}} \quad \Rightarrow \quad \omega = \beta^2 \cdot \sqrt{\frac{EI_y}{\bar{m}}} \tag{32}$$

Thus, inserting equation (31) into equation (32), the eigen-frequency ω_n is directly arise for each n-value.

$$\omega_n = \frac{n^2 \cdot \pi^2}{L^2} \cdot \sqrt{\frac{EI_y}{\bar{m}}} \qquad n = 1,2,3,\dots \tag{33}$$

Therefore, the vibration mode-shape of the examined beam arises by equation (24) -since previous inserting equation (30)- thus:

$$\varphi_n(x) = C_1 \sin \beta x = C_1 \sin \frac{n \cdot \pi \cdot x}{L} \qquad n = 1,2,3,\dots \tag{34}$$

The value of C_1 is arbitrary, and we usually get it equal to unit. Thus, for each value of parameter n, a mode-shape with its eigen-frequency are resulted. The fundamental (first) mode-shape is resulted for $n = 1$, which shows a half sinusoidal wave, the second mode-shape shows a foul sinusoidal wave, etc. (Figure3). The order of the eigen-frequencies are ω_1, $\omega_2 = 4\omega_1$, $\omega_3 = 9\omega_1$, $\omega_4 = 16\omega_1$ etc.

3. The equivalent three degrees of freedom beam

At beams where the fundamental mode-shape does not activate the 90% of the total beam mass, we ask to consider the three first mode-shapes. Thus, for this purpose, we must define an ideal equivalent three degrees of freedom beam, which is going to give the three mode-shapes of the examined beam. Therefore, which is the ideal three degrees of freedom system, where its three mode-shapes coincide with the real first three mode-shapes of the beam with distributed mass and flexural stiffness?

In order to answer the above-mentioned question, consider a weightless beam with length L and constant section along its length, where carry three concentrated masses that each one has the same mass-value m_{eq} , located per distance 0.25L, between one to one, and each one mass possesses a vertical degree of freedom (Fig.4).

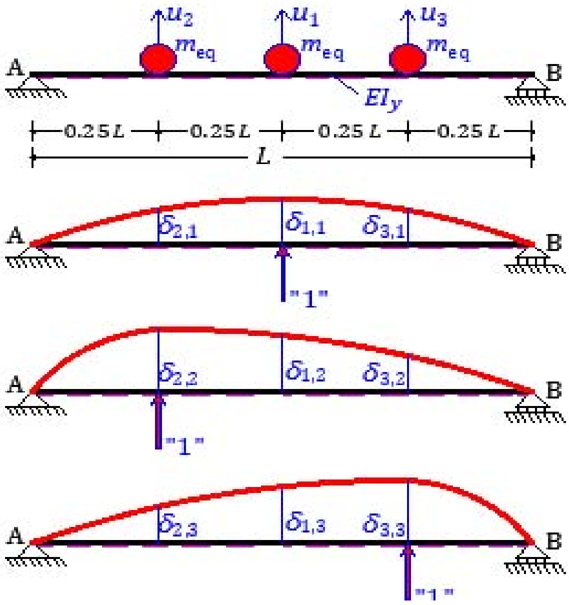

Figure 4. The equivalent three-degree of freedom beam

The beam displacement vector **u** of the three degrees of freedom, as well as the diagonal beam mass matrix $\boldsymbol{m}$ are written:

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \qquad \mathbf{m} = \begin{bmatrix} m_{\text{eq}} & 0 & 0 \\ 0 & m_{\text{eq}} & 0 \\ 0 & 0 & m_{\text{eq}} \end{bmatrix} \tag{35}$$

Furthermore, the beam flexibility matrix $\boldsymbol{f}$ can be calculated using a suitable method (Figure4), and the inverse matrix gives the stiffness matrix $\boldsymbol{k}$ of the three degrees of freedom beam.

$$\boldsymbol{f} = \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & \delta_{1,3} \\ \delta_{2,1} & \delta_{2,2} & \delta_{2,3} \\ \delta_{3,1} & \delta_{3,2} & \delta_{3,3} \end{bmatrix} = \frac{L^3}{48EI_y} \cdot \begin{bmatrix} 1 & 0.6875 & 0.6875 \\ 0.6875 & 0.5625 & 0.4375 \\ 0.6875 & 0.4375 & 0.5625 \end{bmatrix} \tag{36}$$

$$\boldsymbol{k} = \begin{bmatrix} k_{1,1} & k_{1,2} & k_{1,3} \\ k_{2,1} & k_{2,2} & k_{2,3} \\ k_{3,1} & k_{3,2} & k_{3,3} \end{bmatrix} = \frac{48EI_y}{L^3} \cdot \begin{bmatrix} 18.285714 & -12.571429 & -12.571429 \\ -12.571429 & 13.142857 & 5.142857 \\ -12.571429 & 5.142857 & 13.142857 \end{bmatrix} \tag{37}$$

The equations of motion for the case of the free undamped vibration of the ideal beam is given:

$$\mathbf{m}\,\ddot{\mathbf{u}}(t) + \mathbf{k}\,\mathbf{u}(t) = \mathbf{0} \tag{38}$$

The eigen-problem is written:

$$(\mathbf{k} - \boldsymbol{\omega}_n^2 \mathbf{m})\boldsymbol{\varphi}_n = \mathbf{0} \qquad n = 1, 2, 3. \tag{39}$$

where, the eigen-frequencies $\boldsymbol{\omega_n}$ and the three mode-shapes $\boldsymbol{\varphi_n}$ are known by equation (33-34) and Figure 4. Therefore, the unique unknown parameter is the mass m_{eq}. Thus,

$$\det(\mathbf{k} - \boldsymbol{\omega}_1^2 \mathbf{m}) = 0 \qquad \Rightarrow \tag{40}$$

$$m_{\mathrm{eq}}^3 + A \cdot m_{\mathrm{eq}}^2 + B \cdot \mathrm{m}_{\mathrm{eq}} + C = 0 \tag{41}$$

where,

$$A = -\frac{k_{11}+k_{22}+k_{33}}{\omega_1^2}\,, \qquad B = \frac{k_{11}k_{33}+k_{11}k_{22}+k_{22}k_{33}-k_{12}^2-k_{13}^2-k_{23}^2}{\omega_1^4}$$

$$C = -\frac{k_{11}k_{22}k_{33}+2k_{12}k_{13}k_{23}-k_{11}k_{23}^2-k_{22}k_{13}^2-k_{33}k_{12}^2}{\omega_1^6}$$

The numerical solution of equation (41) gives three roots for parameter m_{eq}, where only the first root is acceptable, because the other two values rejected since do not have natural meaning (appear values greater from the total beam mass $\overline{m}L$). Thus, the only one acceptable root is given:

$$m_{eq} = 0.24984748 \cdot (\overline{m}L) \tag{42}$$

Therefore, inserting the ideal equivalent mass m_{eq} by equation (42) at three degrees of freedom system of Figure 4, the three eigen-frequencies and mode-shapes coincide with the real values of the initial beam that has distributed mass and flexural stiffness.

4. Conclusions

The present article has presented a mathematic ideal three degrees of freedom system that is equivalent with the modal behavior of the bi-hinges beam with distributed mass and flexural stiffness along its length. This ideal three degrees of freedom system can be used in instrumentation of a such beam, which does not possess concentrated masses. In the framework of the identification of mode-shapes of an bi-hinges beam (i.e. bay of a bridge), the equivalent mass by equation (24) permits to locate accelerometers per $0.25L$ (as shown at Figure 4) and there measure the response acceleration histories, in order to calculate the real first three mode shapes of the beam.

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